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CONVERGENCE OF POWERS OF CONTROLLABLE INTUITIONISTIC FUZZY MATRICES

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Abstract

Convergences of powers of controllable intuitionistic fuzzy matrices have been studied. It is shown that they oscillate with period equal to 2, in general. Some equalities and sequences of inequalities about powers of controllable intuitionistic fuzzy matrices have been obtained.

Keywords:

Intuitionistic Fuzzy Matrix, Controllable Fuzzy Matrices, Transitivity, Convergence Powers of Intuitionistic Fuzzy Matrix

1. INTRODUCTION

Uncertainty forms have a very important part in our daily life. During the time we handle real life problems involving uncertainty like Medical fields, Engineering, Industry, and Economics and so on. The conventional techniques may not be enough and easy, so Zadeh [1] gave the introduction of fuzzy set theory and this come out to be a gift for the study of some uncertainty types wherever old techniques did not work out. Fuzzy theory and the generalizations regarding it contributed to some remarkable Mathematical applications in so many different problems in real life that involve uncertainties of certain types. Biswas and Ranjit [44] [45] have studied ''Is Fuzzy Theory'' an appropriate tool for large size problems and decision problems in imprecision and uncertainty in information representation and processing. For the motive of handling different types of uncertainties, several generalizations and modifications regarding fuzzy set theory like vague sets, rough sets, soft sets, theory of Intuitionistic Fuzzy Sets (IFSs), and other generalization has been developed too. Among all the above generalization, IFS is most useful. Atanassov [36] [37] developed the concept of IFS. The ideas of IFS were developed later in [38]-[41].

Kim and Rouch [3] introduced the concept of Fuzzy Matrix (FM). FM plays a vital role in various areas in Science and Engineering and solves the problems involving various types of uncertainties [4]. Xin [35] studied some results on the nilpotent FM. Meenakshi [7] studied minus ordering, space ordering and schur complement of FM and block FM. Buckley [34] Ran and Liu [30] and Gregory et al., [33] after applying max-min operation on FM found only two results, either the FM convergences to idempotent matrices or oscillates to finite period. Hashimoto [32] studied the convergence properties of a fuzzy transitive matrix. Later much work has been done by many researchers on FM. FMs deal only with membership value whereas Intuitionistic Fuzzy Matrices (IFMs) deals with both membership and non-membership value. Khan et al. [5] introduced the concept of IFMs and several interesting properties on IFMs have been obtained in [6]. Bhowmik and Pal [11] examined the convergence of max-min powers of an IFM. Pradhan and Pal [20] developed the concept mean powers of convergence of IFMs. Lur et al. [32] studied the powers of convergence of IFMs. Several author's [12]-[19], [21]-[27] worked on IFMs and obtained various interesting results, which are very useful in handling uncertainty problems in our daily life. Xin [8] introduced the notation of controllable FMs. He further, [9] studied convergence of powers of controllable FMs and developed some results on nilpotent FMs. Here we introduce controllable IFMs and develop convergence of powers of controllable IFMs.

2. PRELIMANARIES

Definition 2.1[2] An Intuitionistic Fuzzy Set (IFS) A in X (universal set) is defined as an object of the following form A $=\{\langle x, \mu_A(x), \nu_A(x)\rangle / x\in X\}$, where the functions: $\mu_A: X \to [0, 1]$ and $\nu_A: X\to[0, 1]$ define the membership function and non-membership function of the element $x\in X$ respectively and for every $x\in X$: $0 \le \mu_A(x) + \nu_A(x) \le 1$.

In short we write $\langle x, x'\rangle$ as an intuitionistic fuzzy element with $x + x' \le 1$. For $\langle x, x'\rangle$, $\langle y, y'\rangle \in$ IFS, Atanassov introduced operations $\langle x, x'\rangle \vee \langle y, y'\rangle = \langle max\{x, y\}, min\{x', y'\}\rangle$, $\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle min\{x, y\}, max\{x', y'\}\rangle$, $\langle x, x'\rangle \le \langle y, y'\rangle$ means $x \le y$, $x' \ge y'$ and $\langle x, x'\rangle < \langle y, y'\rangle$ if $x < y$ and $\langle x' < y'\rangle$ in this case we say $\langle x, x'\rangle$, $\langle y, y'\rangle$ are comparable. For any two comparable elements $\langle x, x'\rangle$, $\langle y, y'\rangle \in$ IFS, the operation $\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle$ is defined by

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle = \begin{cases} \langle x, x'\rangle & if\ \langle x, x'\rangle > \langle y, y'\rangle \\ \langle 0,1\rangle & if\ \langle x, x'\rangle \le \langle y, y'\rangle \end{cases} \quad (1)$$

Definition 2.2 [5] Let $X = (x_1, x_2,..., x_m)$ be a set of alternatives and $Y = (y_1, y_2,..., y_n)$ be the attribute set of each element of X. An IFM is defined by A $= (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j)\rangle)$, for $i = 1, 2,...,m$ and $j = 1, 2,...,n$ where $\mu_A: X \times Y \to [0, 1]$ and $\nu_A: X \times Y \to [0, 1]$ satisfy the condition : $0 \le \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \le 1$. For simplicity we denote an IFM is a matrix of pairs A $= (\langle a_{ij}, a_{ij}'\rangle)$ of non-negative real numbers satisfying $a_{ij} + a_{ij}' \le 1$for all i, j. We denote the set of all IFM of order $m \times n$ by mn and n denote the set of IFM of order $n \times n$.

Some of the definitions and results we apply in this paper are given below.

Let $Q = [(\langle q_{ij}, q_{ij}'\rangle]$ and $S = [(\langle s_{ij}, s_{ij}'\rangle]$ be $n \times n$ IFMs with elements in [0, 1].

$$Q \vee S = (\langle q_{ij} \vee s_{ij}, q_{ij}' \vee s_{ij}'\rangle),$$
$$Q \wedge S = (\langle q_{ij} \wedge s_{ij}, q_{ij}' \vee s_{ij}'\rangle),$$
$$Q^c S = (\langle q_{ij}, q_{ij}'\rangle \langle s_{ij}, s_{ij}'\rangle),$$

where,

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle = \begin{cases} \langle x, x'\rangle & if\ \langle x, x'\rangle > \langle y, y'\rangle \\ \langle 0,1\rangle & if\ \langle x, x'\rangle \le \langle y, y'\rangle \end{cases} \quad (2)$$

$Q \times S = [(\langle q_{i1} \wedge s_{1j}, q_{i1}' \vee s_{1j}' \rangle \vee \langle q_{i2} \wedge s_{2j}, q_{i2}' \vee s_{2i}' \rangle \vee, ..., \vee \langle q_{in} \wedge s_{nj}, q_{in}' \vee s_{nj}' \rangle)]$

$$Q^{k+1} = Q^k \times Q,\ k = 1, 2, 3, ...$$

Denote $Q^k = [\langle q_{ij}^k, q_{ij}'^k \rangle]$, $k = 1, 2, 3, ...$

$$\langle (\langle q_{ij}^k, q_{ij}'^k \rangle = \vee^n_{j_{1=1}} \vee^n_{j_{2=1}}, ..., \vee^n_{j_{(k-1)=1}} (q_{ij_1} \wedge q_{j_1 j_2} \wedge ... \wedge q_{j_{k-1} j}) \Lambda^n_{j_{1=1}} \Lambda^n_{j_{2=1}}, ..., \Lambda^n_{j_{(k-1)=1}} (q'_{ij_1} \wedge q'_{j_1 j_2} \wedge ... \wedge q'_{j_{k-1} j}) \rangle$$

$Q^0 = I$, where I is intuitionistic fuzzy unit matrix of order $n \times n$.

$Q^T = [\langle q_{ij}, q_{ij}' \rangle]$, (the transpose)

$$\Delta Q = [\langle q_{ij}, q_{ij}' \rangle]^c [\langle q_{ji}, q_{ji}' \rangle]$$

$$\nabla Q = [\langle q_{ij}, q_{ij}' \rangle] \wedge [\langle q_{ji}, q_{ji}' \rangle]$$

$$Q \le S \text{ iff } (\langle q_{ij}, q_{ij}' \rangle) \le \langle s_{ij}, s_{ij}' \rangle$$

for all $i, j \in 1, 2, .., n$.

The IFM Q is said to be nilpotent if $Q^n = (\langle 0, 1 \rangle)$, max-min transitive if $Q^2 \le Q$ symmetric if $Q = Q^T$; s-transitive iff (for any indices $i, j, k \in 1, 2, ..., n$ with $i \ne j$, $i \ne k$, $j \ne k$, is as $\langle q_{ik}, q_{ik}' \rangle > \langle q_{jk}, q_{jk}' \rangle$, we get $\langle q_{ij}, q_{ij}' \rangle > \langle q_{ji}, q_{ji}' \rangle$. Controllable if there exists a permutation matrix P such that $T = [\langle t_{ji}, t_{ji}' \rangle] = P \times Q \times P^T$ satisfies $\langle t_{ij}, t_{ij}' \rangle \ge \langle t_{ji}, t_{ji}' \rangle$ for $i > j$. Let Q be a nilpotent IFM. If there exists some integer $l (1 \le l \le n)$ such that $Q^{l-1} \ne (\langle 0,1 \rangle)$ and $Q^l = (\langle 0,1 \rangle)$, then l is called the nilpotent length of Q.

Remark 2.3 [46] Let Q, S, H and T be $n \times n$ IFMs, then $Q \times S \vee T = (Q \times S) \vee (Q \times T)$, $(S \vee T) \times Q = (S \times Q) \vee (T \times Q)$, (distributive property). If $Q \le S$, $T \le H$, then $Q \times S \le T \times H$.

Theorem 2.4 [28] Q is controllable IFMs iff $(\Delta Q)^n = (\langle 0, 1 \rangle)$.

Remark 2.5 Symmetric, max-min transitive, s-transitive and nilpotent IFMs are controllable IFMs not vice-versa.

Theorem 2.6 [26] If Q is max-min transitive IFM then, $Q^n = Q^{n+1}$.

Theorem 2.7 [29] If Q is s-transitive IFM then,

(i) $Q^{3n-4} = Q^{3n-2}$

(ii) $Q^{3n-3} = Q^{3n-1}$

Definition 2.8 [10] A $m \times n$ zero IFM 'O' is a matrix, whose entries are $\langle 0,1 \rangle$. The $n \times n$ identity IFM 'I_n' is defined by $(\langle \delta_{ij}, \delta_{ij}' \rangle)$ such that $\delta_{ij} = 1$, $\delta_{ij}' = 0$, if $i = j$ and $\delta_{ij} = 0$, $\delta_{ij}' = 1$, if $i \ne j$.

Definition 2.9[10] A square IFM is called permutation IFM, if every row and column contains exactly one $\langle 0, 1 \rangle$ and other all entries are $\langle 0,1 \rangle$. Let P_n be the set of all $m \times n$ such matrices in n. If $A \in P_n$, then $AA^T = A^T A = I_n$, A^T is the transpose of A.

3. PROPERTIES OF CONTROLLABLE INTUITIONISTIC FUZZY MATRIX

If Q is a controllable IFM (then $(\Delta Q)^n = \langle 0, 1 \rangle$ by Theorem 2.4) the nilpotent length of ΔQ is called controllable degree of Q, denoted by $c(Q)$. So, it is clear $1 \le c(Q) \le n$.

Put $Q^k = [\langle q_{ij}^k, q_{ij}'^k \rangle]$, $k = 1, 2, 3, ...$, $\Delta Q = [\langle q_{ij}^\Delta, q_{ij}'^\Delta \rangle]$ and $(\Delta Q)^k = [\langle q_{ij}^{\Delta,k}, q_{ij}'^{\Delta,k} \rangle]$, $k = 1, 2, 3, ...$

Theorem 3.1 If Q is controllable IFM then,

(1) $$Q^{l+0} \le Q^{l+2} \le Q^{l+4} \le Q^{l+6} \le ..., \quad (3)$$

(2) $$Q^{l+1} \le Q^{l+3} \le Q^{l+5} \le Q^{l+7} \le ..., \quad (4)$$

where, $l = c(Q)$.

Proof: $Q^{l+0} \le Q^{l+2} \le Q^{l+4} \le Q^{l+6} \le ...,$

Let, $\langle q_{ij}^l, q_{ij}'^l \rangle = \langle q_{h_0 h_1}, q'_{h_0 h_1} \rangle \wedge \langle q_{h_1 h_2}, q'_{h_1 h_2} \rangle \wedge ... \wedge \langle q_{h_{l-1} h_l}, q'_{h_{l-1} h_l} \rangle$ where $i = h_0$, $j = h_l$.

If $\langle q_{h_s h_{s+1}}, q'_{h_s h_{h+1}} \rangle > \langle q_{h_{s+1} h_s}, q'_{h_{s+1} h_s} \rangle$ would hold for each $s = 0, 1, 2, ..., l-1$ then, $\langle q^\Delta_{h_s h_{s+1}}, q'^\Delta_{h_s h_{s+1}} \rangle > \langle 0,1 \rangle$.

Hence,

$$\langle q_{ij}^{\Delta,l}, q_{ij}'^{\Delta,l} \rangle \ge \langle q^\Delta_{h_0 h_1}, q'^\Delta_{h_0 h_1} \rangle \wedge \langle q^\Delta_{h_1 h_2}, q'^\Delta_{h_1 h_2} \rangle \wedge ... \wedge \langle q^\Delta_{h_{l-1} h_l}, q'^\Delta_{h_{l-1} h_l} \rangle > \langle 0,1 \rangle$$

which contradicts with the fact that,

$$(\Delta Q)^l = \langle q_{ij}^{\Delta,l}, q_{ij}'^{\Delta,l} \rangle = \langle 0, 1 \rangle.$$

Thus $\langle q_{h_s h_{s+1}}, q'_{h_s h_{h+1}} \rangle \le \langle q_{h_{s+1} h_s}, q'_{h_{s+1} h_s} \rangle$ for at least one $s \in \{0, 1, 2, ..., l-1\}$ and

$$\langle q_{ij}^l, q_{ij}'^l \rangle = \langle q_{h_0 h_1}, q'_{h_0 h_1} \rangle \wedge \langle q_{h_1 h_2}, q'_{h_1 h_2} \rangle \wedge ... \wedge \langle q_{h_s h_{s+1}}, q'_{h_s h_{s+1}} \rangle \wedge \langle q_{h_{s+1} h_s}, q'_{h_{s+1} h_s} \rangle \wedge \langle q_{h_s h_{s+1}}, q'_{h_s h_{s+1}} \rangle \wedge \langle q_{h_{s+1} h_{s+2}}, q'_{h_{s+1} h_{s+2}} \rangle ... \wedge \langle q_{h_{l-1} h_l}, q'_{h_{l-1} h_l} \rangle \le \langle q_{ij}^{l+2}, q_{ij}'^{l+2} \rangle$$

So, $Q^l \le Q^{l+2}$.

Then, $Q^{l+2} = Q^l \times Q^2 \le Q^{l+2} \times Q^2 = Q^{l+4}$ (by Remark 2.3).

$Q^{l+4} = Q^{l+2} \times Q^2 \le Q^{l+4} \times Q^2 = Q^{l+6}$.

Hence,

$$Q^{l+0} \le Q^{l+2} \le Q^{l+4} \le Q^{l+6} \le ..., \quad (2)$$

From Eq.(1) we have $Q^1 \le Q^{l+2}$. Then,

$$Q^{l+1} = Q^l \times Q \le Q^{l+2} \times Q = Q^{l+3},$$

$$Q^{l+3} = Q^{l+1} \times Q^2 \le Q^{l+3} \times Q^2 = Q^{l+5}, ...$$

Thus,

$$Q^{l+1} \le Q^{l+3} \le Q^{l+5} \le Q^{l+7} \le ...$$

Theorem 2.2 Let Q be $n \times n$ IFM and let $B = Q \vee I$ (I is the unit IFM). Then,

$$B \le B^2 \le ... \le B^{n-1} = B^n = B^{n+1} = ...$$

Proof:

We shall show a stronger result which claims that there exists an integer $m \le n-1$ such that

$$B \le B^2 \le ... \le B^m = B^{m+1} = ...$$

Let, $B = (\langle a_{ij}, a'_{ij} \rangle)$ and $B^h = (\langle b_{ij}, b'_{ij} \rangle)$.

Then,

$$B^{h+1} = \left(\sum_{k=1}^{n} \langle b_{ik}, b'_{ik} \rangle \langle a_{kj}, a'_{kj} \rangle \right)$$

$$= \left(\langle b_{ij}, b'_{ij} \rangle + \sum_{k=1, k \ne j}^{n} \langle b_{ik}, b'_{ik} \rangle \langle a_{kj}, a'_{kj} \rangle \right)$$

Since, $\langle a_{jj}, a_{jj}' \rangle = \langle 1, 0 \rangle$. Thus

$$\langle b_{ij}, b'_{ij} \rangle \le \sum_{k=1}^{n} \langle b_{ik}, b'_{ik} \rangle \langle a_{kj}, a'_{kj} \rangle$$

and

$$B^h = B^{h+1}.$$

This proves that $B \le B^2 \le ...$ we must prove that this chain does not strictly increase indefinitely. Then we have to show that $B^{n-1} = B^n$. Since already we know that $B^{n-1} \le B^n$. We need to show that $B^n \le B^{n-1}$. We are using the fact $\le$ is a partial ordering, and hence it is anti-symmetric.

Consider an off-diagonal element of B^n. It is of the following form

$$\sum_{k_{n-1}=1}^{n} \sum_{k_{n-2}=1}^{n} \cdots \sum_{k_1=1}^{n} \begin{matrix} \langle a_{ik_1}, a'_{ik_1} \rangle \langle a_{k_1k_2}, a'_{k_1k_2} \rangle ... \\ \langle a_{k_{n-2}k_{n-1}}, a'_{k_{n-2}k_{n-1}} \rangle \langle a_{k_{n-1}j}, a'_{k_{n-1}j} \rangle \end{matrix}$$

Clearly, there are n-1 + 2 = n + 1 subscripts, so not all of them can be distinct.

Consider a term of the above form and suppose there exists an integer s such that $j = k$. Then term is of the form:

$\langle a_{ik_1}, a'_{ik_1} \rangle \langle a_{k_1k_2}, a'_{k_1k_2} \rangle ... \langle a_{k_{s-1}j}, a'_{k_{s-1}j} \rangle \langle a_{jk_{s+1}}, a'_{jk_{s+1}} \rangle ... \langle a_{k_{n-1}j},$ $a'_{k_{n-1}j} \rangle$, but since $\langle a, a' \rangle \langle b, b' \rangle \le \langle a, a' \rangle$, is contained in the term

$$\langle a_{ik_1}, a'_{ik_1} \rangle ... \langle a_{k_{s-1}j}, a'_{k_{s-1}j} \rangle ,$$

which is the $(i, j)^{th}$ entry of B^S and is contained in the $(i, j)^{th}$ entry of B^{n-1}. The previous argument is symmetric if $i = k_s$, for some integer s. The remaining case occurs if there exist integer r and s such that $k_s = k_r$. Assuming $s < r$ we get,

$$\begin{gathered} \langle a_{ik_1}, a'_{ik_1} \rangle ... \langle a_{k_{s-1}k_r}, a'_{k_{s-1}k_r} \rangle \\ \langle a_{k_rk_{s+1}}, a'_{k_rk_{s+1}} \rangle ... \langle a_{k_{r-1}k_r}, a'_{k_{r-1}k_r} \rangle \\ \langle a_{k_rk_{r+1}}, a'_{k_rk_{r+1}} \rangle ... \langle a_{k_{n-1}j}, a'_{k_{n-1}j} \rangle, \end{gathered}$$

is contained in

$$\begin{gathered} \langle a_{ik_1}, a'_{ik_1} \rangle ... \langle a_{k_{s-1}k_r}, a'_{k_{s-1}k_r} \rangle \langle a_{k_rk_{r+1}}, a'_{k_rk_{r+1}} \rangle \\ \langle a_{k_rk_{s+1}}, a'_{k_rk_{s+1}} \rangle ... \langle a_{k_{n-1}j}, a'_{k_{n-1}j} \rangle. \end{gathered}$$

This term is contained in the $(i, j)^{th}$ entry of B^{n-1}. Therefore $B^n \le B^{n-1} \Rightarrow B^n = B^{n-1}$.

Theorem 3.3 If Q is controllable IFM then $Q^{n+1} \le Q^{n+l-1} \vee Q^{n+l-2}$, where $l = c(Q)$.

Proof:

By doing mathematical induction and by Remark 2.3; we can easily get,

for k=1

$$(I \vee Q) = I \vee Q ,$$

for k=2

$$\begin{aligned} (I \vee Q)^2 &= (I \vee Q) \times (I \vee Q) = ((I \vee Q) \times I) \vee ((I \vee Q) \times Q) \\ &= [(I \times I) \vee (I \times Q)] \vee [(I \times Q) \vee (Q \times Q)] \\ &= (I \vee Q) \vee (Q \vee Q) \\ &= I \vee Q \vee Q^2 \end{aligned}$$

$$(I \vee Q)^k = I \vee Q \vee Q^2 \vee ... \vee Q^{k-1} \vee Q^k, k = 3,4...$$

Then,

$$(I \vee Q)^{n-1} = I \vee Q \vee Q^2 \vee ... \vee Q^{n-2} \vee Q^{n-1} \quad (5)$$

$$(I \vee Q)^n = I \vee Q \vee Q^2 \vee ... \vee Q^{n-1} \vee Q^n \quad (6)$$

Since $(I \vee Q)^{n-1} = (I \vee Q)^n$ by Theorem 3.2 we get $Q^l \times (I \vee Q)^{n-1} = Q^l \times (I \vee Q)^n$, that is,

$$\begin{aligned} & Q^l \vee Q^{l+1} \vee Q^{l+2} \vee ... \vee Q^{n+l-2} \vee Q^{n+l-1} \\ &= Q^l \vee Q^{l+1} \vee Q^{l+2} \vee ... \vee Q^{n+l-1} \vee Q^{n+1} \end{aligned} \quad (7)$$

using Theorem 3.1 in Eq.(6) we get,

$Q^{n+l-2} \vee Q^{n+l-1} = Q^{n+l-1} \vee Q^{n+l}$, so, $Q^{n+l} \le Q^{n+l-2} \vee Q^{n+l-1}$.

Theorem 3.4 If Q is controllable IFM and $c(Q) = l \ge 2$, then

(1) $Q^{2n+l-4} = Q^{2n+l-2}$ (8)

(2) $Q^{2n+l-3} = Q^{2n+l-1}$ (9)

Proof:

(1) By Theorem 3.1

$$\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+1-2} \rangle \ge \langle q_{ij}^{2n+l-4}, q_{ij}'^{2n+1-4} \rangle .$$

It is sufficient to prove that

$$\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+1-2} \rangle \le \langle q_{ij}^{2n+l-4}, q_{ij}'^{2n+1-4} \rangle .$$

Let

$$\begin{aligned} & \langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+1-2} \rangle = \\ & \langle q_{h_0h_1}, q'_{h_0h_1} \rangle \wedge \langle q_{h_1h_2}, q'_{h_1h_2} \rangle \wedge ... \wedge \langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n} \rangle \wedge \\ & \langle q_{h_nh_{n+1}}, q'_{h_nh_{n+1}} \rangle \wedge ... \wedge \langle q_{h_{n+l-3}h_{n+l-2}}, q'_{h_{n+l-3}h_{n+l-2}} \rangle \wedge \\ & \langle q_{h_{n+l-2}h_{n+l-1}}, q'_{h_{n+l-2}h_{n+l-1}} \rangle \wedge \langle q_{h_{n+l-1}h_{n+l}}, q'_{h_{n+l-1}h_{n+l}} \rangle \wedge \\ & \langle q_{h_{n+l}h_{n+l+1}}, q'_{h_{n+l}h_{n+l+1}} \rangle \wedge ... \wedge \langle q_{h_{2n+l-3}h_{2n+l-2}}, q'_{h_{2n+l-3}h_{2n+l-2}} \rangle \end{aligned} \quad (10)$$

$h_0 = i$ and $h_{2n+l-2} = j$. Note that if l = 2, the Eq.(10) becomes as

$$\begin{aligned} \langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+1-2} \rangle &= \langle q_{ij}^{2n}, q_{ij}'^{2n} \rangle = \langle q_{h_0h_1}, q'_{h_0h_1} \rangle \wedge \langle q_{h_1h_2}, q'_{h_1h_2} \rangle \\ & \wedge ... \wedge \langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n} \rangle \wedge \langle q_{h_nh_{n+1}}, q'_{h_nh_{n+1}} \rangle \\ & \wedge ... \wedge \langle q_{h_{2n-1}h_{2n}}, q'_{h_{2n-1}h_{2n}} \rangle \end{aligned} \quad (11)$$

We have $h_a = h_b$ for at least one $a, b \in \{0, 1, 2,...,n\} (a<b)$. Since $h_t \in \{0, 1, 2, 3,...,n\}$ for every t = 0, 1, 2,..., n. Similarly, we have $h_c = h_d$ (c < d) for at least one $c, d \in \{n+l-2, n+2-l, n+l, n+l+1, n+l+2,...,2,+l-2\}$.

Let, $p = b-a$, $v = d-c$, p and v be the elements in sets.

$$\{\langle q_{h_ah_{a+1}}, q'_{h_ah_{a+1}} \rangle, \langle q_{h_{a+1}h_{a+2}}, q'_{h_{a+1}h_{a+2}} \rangle, ..., \langle q_{h_{b-1}h_b}, q'_{h_{b-1}h_b} \rangle\}$$

and

$$\{\langle q_{h_ch_{c+1}}, q'_{h_ch_{c+1}} \rangle, \langle q_{h_{c+1}h_{c+2}}, q'_{h_{c+1}h_{c+2}} \rangle, ..., \langle q_{h_{d-1}h_d}, q'_{h_{d-1}h_d} \rangle\}$$

respectively.

Two cases arise

(I) Among p and v at least one number is even say p. Eliminating

$\{\langle q_{h_a h_{a+1}}, q'_{h_a h_{a+1}}\rangle \wedge \langle q_{h_{a+1}h_{a+2}} \wedge q'_{h_{a+1}h_{a+2}}\rangle \wedge \ldots \wedge \langle q_{h_{b-1}h_b}, q'_{h_{b-1}h_b}\rangle\}$ from Eq.(10) and now using Theorem 3.1 we get a sequence of inequalities as

$$\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+l-2}\rangle \le \langle q_{ij}^{2n+l-2-p}, q_{ij}'^{2n+l-2-p}\rangle \le$$
$$\langle q_{ij}^{2n+l-2-p+2}, q_{ij}'^{2n+l-2-p+2}\rangle \le \ldots \le \langle q_{ij}^{2n+l-4}, q_{ij}'^{2n+l-4}\rangle$$

since $p \le n$ and $2n + l\text{-}2\text{-}p \ge l$.

(II) Among p and v no one is even. But sum of them is even number.

Sub cases:

(II) (a) n is even number.

Eliminating

$$\{\langle q_{h_a h_{a+1}}, q'_{h_a h_{a+1}}\rangle \wedge \langle q_{h_{a+1}h_{a+2}} \wedge q'_{h_{a+1}h_{a+2}}\rangle \wedge \ldots \wedge \langle q_{h_{b-1}h_b}, q'_{h_{b-1}h_b}\rangle\}$$

and

$$\{\langle q_{h_c h_{c+1}}, q'_{h_c h_{c+1}}\rangle \wedge \langle q_{h_{c+1}h_{c+2}} \wedge q'_{h_{c+1}h_{c+2}}\rangle \wedge \ldots \wedge \langle q_{h_{d-1}h_d}, q'_{h_{d-1}h_d}\rangle\}$$

from Eq.(10) and using Theorem 3.1, we get

$$\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+l-2}\rangle \le \langle q_{ij}^{2n+l-2-p-v}, q_{ij}'^{2n+l-2-p-v}\rangle$$
$$\le \langle q_{ij}^{2n+l-2-p-v}, q_{ij}'^{2n+l-2-p-v}\rangle \le \ldots \le \langle q_{ij}^{2n+l-4}, q_{ij}'^{2n+l-4}\rangle,$$

since then $p + v \le (n\text{-}1) + (n\text{-}1) \le 2n\text{-}2$ and $2n + l\text{-}2\text{-}(p+v) \ge l$.

(II) (b) n is an odd number. If among p and v at least one is less than n, then $p + v \le n + (n\text{-}2) \le 2n\text{-}2$ and $2n + l\text{-}2\text{-}(p+q) \ge l$, its proof is same as (II) (a).

Now consider the case: $p = n$ and $v = n$. In this case two sub cases arise

(1) Let $l \ge 3$, if

$\left\langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n}\right\rangle > \left\langle q_{h_n h_{n-1}}, q'_{h_n h_{n-1}}\right\rangle$ and

$\left\langle q_{h_{a+l-2}h_{n+l-1}}, q'_{h_{a+l-2}h_{n+l-1}}\right\rangle > \left\langle q_{h_{n+l-1}h_{a+l-2}}, q'_{h_{n+l-1}h_{a+l-2}}\right\rangle$,

then, we have $\left\langle q_{h_s h_{s+1}}, q'_{h_s h_{s+1}}\right\rangle \le \left\langle q_{h_{s+1}h_s}, q'_{h_{s+1}h_s}\right\rangle$ for at least one $s \in \{n, n+1, \ldots, n+l-3\}$. (In fact, if $\left\langle q_{h_s h_{s+1}}, q'_{h_s h_{s+1}}\right\rangle > \left\langle q_{h_{s+1}h_s}, q'_{h_{s+1}h_s}\right\rangle$ it would hold for each $s = n+1, \ldots, n+l-3$), then we get

$$\left\langle q^{\Delta}_{h_s h_{s+1}}, q'^{\Delta}_{h_s h_{s+1}}\right\rangle = \left\langle q_{h_s h_{s+1}}, q'_{h_s h_{s+1}}\right\rangle \leftharpoondown \left\langle q_{h_{s+1}h_s}, q'_{h_{s+1}h_s}\right\rangle$$
$$= \left\langle q_{h_s h_{s+1}}, q'_{h_s h_{s+1}}\right\rangle > \langle 0,1\rangle$$

$s = n\text{-}1, n, n+1, \ldots, n+p\text{-}2$. Then,

$$\left\langle q^{\Delta l}_{h_n h_{n+l-1}}, q'^{\Delta l}_{h_n h_{n+l-1}}\right\rangle \ge \left\langle q^{\Delta}_{h_{n-1}h_n}, q'^{\Delta}_{h_{n-1}h_n}\right\rangle \wedge \left\langle q^{\Delta}_{h_n h_{n+l}}, q'^{\Delta}_{h_n h_{n+l}}\right\rangle \wedge, \ldots \wedge \left\langle q^{\Delta}_{h_{n+l-2}h_{n+l-1}}, q'^{\Delta}_{h_{n+l-2}h_{n+l-1}}\right\rangle > \langle 0,1\rangle.$$

Contradicts with the fact $(\Delta R)^l = \left[\left\langle q_{ij}^{\Delta l}, q_{ij}'^{\Delta l}\right\rangle\right] = (\langle 0,1\rangle)$

Eliminating

$$\left\langle q_{h_0 h_1}, q'_{h_0 h_1}\right\rangle \wedge \left\langle q_{h_1 h_2}, q'_{h_1}\right\rangle \wedge \ldots \wedge \left\langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n}\right\rangle$$

and

$$\left\langle q_{h_{n+l-2}h_{n+l-1}}, q'_{h_{n+l-2}h_{n+l-1}}\right\rangle \wedge \left\langle q_{h_{n+l-1}h_{n+l}}, q'_{h_{n+l-1}h_{n+l}}\right\rangle$$
$$\wedge \ldots \wedge \left\langle q_{h_{2n+l-3}h_{2n+l-2}}, q'_{h_{2n+l-3}h_{2n+l-2}}\right\rangle$$

from Eq.(10), inserting $\left\langle q_{h_{s+1}h_s}, q'_{h_{s+1}h_s}\right\rangle \wedge \left\langle q_{h_s h_{s+1}}, q'_{h_s h_{s+1}}\right\rangle$ into Eq.(10) and apply Theorem 3.1, we arrive at

$$\left\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+l-2}\right\rangle \le \left\langle q_{ij}^{2n+l-2n+2}, q_{ij}'^{2n+l-2n+2}\right\rangle$$
$$\le \left\langle q_{ij}^{l}, q_{ij}'^{l}\right\rangle \le \ldots \le \left\langle q_{ij}^{2n+l-4}, q_{ij}'^{2n+l-4}\right\rangle$$

If $\left\langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n}\right\rangle \le \left\langle q_{h_n h_{n-1}}, q'_{h_n h_{n-1}}\right\rangle$ or

$\left\langle q_{h_{n+l-2}h_{n+l-1}}, q'_{h_{n+l-2}h_{n+l-1}}\right\rangle \le \left\langle q_{h_{n+l-1}h_{n+l-2}}, q'_{h_{n+l-1}h_{n+l-2}}\right\rangle$

say $\left\langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n}\right\rangle \le \left\langle q_{h_n h_{n-1}}, q'_{h_n h_{n-1}}\right\rangle$

then eradicate $\left\langle q_{h_0 h_1}, q'_{h_0 h_1}\right\rangle \wedge \left\langle q_{h_1 h_2}, q'_{h_1}\right\rangle \wedge \ldots \wedge \left\langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n}\right\rangle$, and

$$\left\langle q_{h_{n+l-2}h_{n+l-1}}, q'_{h_{n+l-2}h_{n+l-1}}\right\rangle \wedge \left\langle q_{h_{n+l-1}h_{n+l}}, q'_{h_{n+l-1}h_{n+l}}\right\rangle$$
$$\wedge \ldots \wedge \left\langle q_{h_{2n+l-3}h_{2n+l-2}}, q'_{h_{2n+l-3}h_{2n+l-2}}\right\rangle,$$

from Eq.(10), inserting $\left\langle q_{h_n h_{n-1}}, q'_{h_n h_{n-1}}\right\rangle \wedge \left\langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n}\right\rangle$ into Eq.(10) and applying Theorem 3.1, we can write down (at this point, $i = h_n$, $j = h_{n+l-2}$)

$$\left\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+l-2}\right\rangle \le \left\langle q_{h_n h_{n-1}}, q'_{h_n h_{n-1}}\right\rangle \wedge \left\langle q_{h_{n-1}h_n}, q'_{h_{n-1}h_n}\right\rangle \wedge$$
$$\left\langle q_{h_n h_{n+1}}, q'_{h_n h_{n+1}}\right\rangle \wedge \left\langle q_{h_{n+1}h_{n+2}}, q'_{h_{n+1}h_{n+2}}\right\rangle \wedge \ldots \wedge \left\langle q_{h_{n-l-3}h_{n-l-2}}, q'_{h_{n-l-3}h_{n-l-2}}\right\rangle$$
$$\left\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+l-2}\right\rangle \le \left\langle q_{ij}^{l}, q_{ij}'^{l}\right\rangle \le \left\langle q_{ij}^{l+2}, q_{ij}'^{l+2}\right\rangle \le \ldots \le \left\langle q_{ij}^{2n+l-4}, q_{ij}'^{2n+l-4}\right\rangle$$

(2) $l = 2$ come across at Eq.(11). We have $p = v = n$ then $i = h_0 = h_n = h_{2n} = j$, thus

$$\left\langle q_{ij}^{2n+l-2}, q_{ij}'^{2n+l-2}\right\rangle = \left\langle q_{ii}^{2n}, q_{ii}'^{2n}\right\rangle =$$
$$\left\langle q_{ih_1}, q'_{ih_1}\right\rangle \wedge \left\langle q_{h_1 h_2}, q'_{h_1 h_2}\right\rangle \wedge \ldots \wedge \left\langle q_{h_{n-1}i}, q'_{h_{n-1}i}\right\rangle \wedge \left\langle q_{ih_{n+1}}, q'_{ih_{n+1}}\right\rangle \wedge \left\langle q_{h_{n+1}h_{n+2}}, q'_{h_{n+1}h_{n+2}}\right\rangle \wedge \ldots \wedge \left\langle q_{h_{2n-1}i}, q'_{h_{2n-1}i}\right\rangle \quad (12)$$

If,

$$\left\langle q_{h_{n-1}i}, q'_{h_{n-1}i}\right\rangle > \left\langle q_{ih_{n-1}}, q'_{ih_{n-1}}\right\rangle \;\&\; \left\langle q_{ih_{n+1}}, q'_{ih_{n+1}}\right\rangle > \left\langle q_{h_{n+1}i}, q'_{h_{n+1}i}\right\rangle \quad (13)$$

then $\left\langle q^{\Delta}_{h_{n-1}i}, q'^{\Delta}_{h_{n-1}i}\right\rangle > \langle 0,1\rangle$, $\left\langle q^{\Delta}_{ih_{n+1}}, q'^{\Delta}_{ih_{n+1}}\right\rangle > \langle 0,1\rangle$,

thus $\left\langle q^{\Delta,2}_{h_{n-1}h_{n+1}}, q'^{\Delta,2}_{h_{n-1}h_{n+1}}\right\rangle \ge \left\langle q^{\Delta}_{h_{n-1}i}, q'^{\Delta}_{h_{n-1}i}\right\rangle \wedge \left\langle q^{\Delta}_{ih_{n+1}}, q'^{\Delta}_{ih_{n+1}}\right\rangle > \langle 0,1\rangle$

which contradicts $(\Delta R)^2 = \left[\left\langle q_{ij}^{\Delta,2}, q_{ij}'^{\Delta,2}\right\rangle\right] = \langle 0,1\rangle$.

Thus

$\left\langle q_{h_{n-1}i}, q'_{h_{n-1}i}\right\rangle \le \left\langle q_{ih_{n-1}}, q'_{ih_{n-1}}\right\rangle$ or $\left\langle q_{ih_{n+1}}, q'_{ih_{n+1}}\right\rangle \le \left\langle q_{h_{n+1}i}, q'_{h_{n+1}i}\right\rangle$,

say $\left\langle q_{h_{n-1}i}, q'_{h_{n-1}i}\right\rangle \le \left\langle q_{ih_{n-1}}, q'_{ih_{n-1}}\right\rangle$.

Putting $\left\langle q_{ih_{n-1}}, q'_{ih_{n-1}} \right\rangle \wedge \left\langle q_{h_{n-1}i}, q'_{h_{n-1}i} \right\rangle$ into Eq.(12) and eradicate

$$\left\langle q_{ih_1}, q'_{ih_1} \right\rangle \wedge \left\langle q_{h_1h_2}, q'_{h_1h_2} \right\rangle \wedge \ldots \wedge \left\langle q_{h_{n-1}i}, q'_{h_{n-1}i} \right\rangle$$

and

$$\left\langle q_{ih_{n+1}}, q'_{ih_{n+1}} \right\rangle \wedge \left\langle q_{h_{n+1}h_{n+2}}, q'_{h_{n+1}h_{n+2}} \right\rangle \wedge \ldots \wedge \left\langle q_{h_{2n-1}i}, q'_{h_{2n-1}i} \right\rangle,$$

we obtain,

$$\left\langle q_{ii}^{2n}, q_{ii}'^{2n} \right\rangle \le \left\langle q_{ih_{n-1}}, q'_{ih_{n-1}} \right\rangle \wedge \left\langle q_{h_{n-1}i}, q'_{h_{n-1}i} \right\rangle \le \left\langle q_{ii}^{2}, q_{ii}'^{2} \right\rangle$$

$$= \left\langle q_{ii}^{l}, q_{ii}'^{l} \right\rangle \le \ldots \le \left\langle q_{ii}^{2n-2}, q_{ii}'^{2n-2} \right\rangle = \left\langle q_{ii}^{2n+l-4}, q_{ii}'^{2n+-4} \right\rangle.$$

The proof of Eq.(2) directly follows from Eq.(1).

Theorem 3.5 If Q is symmetric IFM then,

(i) $Q^{2n-2} = Q^{2n}$

(ii) $Q \le Q^3 \le Q^5 \le Q^6$ (14)

Proof:

(i) Proof directly follows Theorem 3.4 Eq.(1), when $l = 2$.

(ii) Since $c(Q) = 1$ in Theorem 3.1 we get (ii)

Example 3.6

$$Q = \begin{bmatrix} \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle & \langle 0.2,0.5\rangle \\ \langle 0.6,0.2\rangle & \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle \\ \langle 0.2,0.5\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \end{bmatrix} \text{is symmetric matrix}$$

$$Q^2 = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle \\ \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \end{bmatrix}$$

$$Q^3 = \begin{bmatrix} \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \end{bmatrix}$$

$$Q^4 = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \end{bmatrix}$$

$$Q^5 = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.2\rangle \end{bmatrix}$$

Q is symmetric, $Q^{2n-3} = Q^3 \ne Q^5 = Q^{2n-1}$. Hence the conclusion of Theorem 3.5 is accurate in general.

Let Q be a max -min transitive IFM. From Theorem 2.6 we get $Q^{n+1} = Q^n$. Now, we shall show that a stronger result which claims that there exists an integer $l \le n$ such that $Q^{l+1} = Q^l$.

Corollary 3.7 Let Q be max-min transitive IFM then, $Q^{l+1} = Q^l$, where $c(Q) = l$.

Proof:

Since Q is max-min transitive IFM then we have

$$Q \ge Q^2 \ge Q^3 \ge \ldots \ge Q^l \ge Q^{l+1} \ge Q^{l+2} \ge \ldots$$

On the other hand, by Theorem 3.1, we have $Q^l \le Q^{l+2}$. Thus $Q^l = Q^{l+2}$, but $Q^l \ge Q^{l+1} \ge Q^{l+2}$, so $Q^l = Q^{l+1}$.

Let Q be s-transitive. Then $c(Q) = l \le n$. By applying Theorem 3.4, we can refine Theorem 2.7 as follows.

Corollary 3.8 If Q is s-transitive then

(1) $Q^{2n+l-4} = Q^{2n+l-2}$,

(2) $Q^{2n+l-3} = Q^{2n+l-1}$, where $c(Q_2) = 1$.

Example 3.9

$$Q_1 = \begin{bmatrix} \langle 0.0,1.0\rangle & \langle 0.8,0.2\rangle & \langle 0.1,0.8\rangle & \langle 0.1,0.8\rangle & \langle 0.7,0.3\rangle \\ \langle 0.8,0.2\rangle & \langle 0.0,1.0\rangle & \langle 0.8,0.2\rangle & \langle 0.1,0.8\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.8,0.2\rangle & \langle 0.0,1.0\rangle & \langle 0.8,0.2\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.8,0.2\rangle & \langle 0.0,1.0\rangle & \langle 0.8,0.2\rangle \\ \langle 0.7,0.3\rangle & \langle 0.0,1.0\rangle & \langle 0.1,0.8\rangle & \langle 0.9,0.1\rangle & \langle 0.0,1.0\rangle \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} \langle 0.0,1.0\rangle & \langle 0.7,0.3\rangle & \langle 0.1,0.8\rangle & \langle 0.2,0.7\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.8,0.2\rangle & \langle 0.1,0.8\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.6,0.3\rangle & \langle 0.0,1.0\rangle \\ \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.0,1.0\rangle & \langle 0.5,0.4\rangle \\ \langle 0.8,0.2\rangle & \langle 0.1,0.8\rangle & \langle 0.2,0.7\rangle & \langle 0.9,0.1\rangle & \langle 0.0,1.0\rangle \end{bmatrix}$$

it can be easily shown that Q_1 is s-transitive, by using the method developed in [29] we can verify whether an IFM is *s*-transitive or not. Let $c(Q_1) = 2$ applying Corollary 3.7 we would have, $Q_1^8 = Q_1^{2n+2-4} = Q_1^{2n+2-2} = Q_1^{10}$. It is easy to prove that $Q_1^8 = Q_1^{10}$.

Q_2 is controllable, but not s-transitive, by using the method developed in [28] we can verify whether an IFM is controllable or not. Let $c(Q_2) = 5$, $Q_2 \ne Q_2$ for all $k, m \in \{1, 2, 3, \ldots, 11\}$ $(k \ne m)$ and $Q_2^{13} = Q_2^{2n+l-4} = Q_2^{2n+l-2} = Q_2^{13}$.

4. CONCLUSION

In this paper we have defined controllable intuitionistic fuzzy matrix and convergence properties for controllable intuitionistic fuzzy matrices are explored. Further, some equalities and sequences of inequalities about powers of controllable intuitionistic fuzzy matrices are obtained.

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